1880MNRAS..41...18G

Addition to a Paper on the Method of Least Squares (vol. xl. pp. 600–614). By J. W. L. Glaisher, M.A., F.R.S.

§ 27. The expressions given in § 14 for ϵ_x^2, ϵ_y^2, &c., the squares of the mean errors of x_0, y_0, &c., admit of being transformed by means of the following theorem:—

If D denote any determinant, and if $D_{m,p}$ denote the determinant formed from it by omitting the m^{th} column and the p^{th} row, and if $D_{mn,pq}$ denote that formed by omitting the m^{th} and n^{th} columns and the p^{th} and q^{th} rows, then

$$DD_{mn,pq} = D_{m,p}\,D_{n,q} - D_{m,q}\,D_{n,p}\,.$$

Applying the theorem to the numerator of ϵ_x^2, and taking the m^{th} and n^{th} columns to be the first and last, and the p^{th} and q^{th} rows to be the first and last, we obtain the formula

$$\begin{vmatrix} (bb), & (bc), & \ldots & (bf) \\ (cb), & (cc), & \ldots & (cf) \\ \cdot & \cdot & \cdot & \cdot \\ (fb), & (fc), & \ldots & (ff) \end{vmatrix} \begin{vmatrix} (aa), & (ab), & \ldots & (af) & (an) \\ (ba), & (bb), & \ldots & (bf), & (bn) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ (fa), & (fb), & \ldots & (ff), & (fn) \\ (na), & (nb), & \ldots & (nf), & (nn) \end{vmatrix}$$

$$= \begin{vmatrix} (aa), & (ab), & \ldots & (af) \\ (ba), & (bb), & \ldots & (bf) \\ \cdot & \cdot & \cdot & \cdot \\ (fa), & (fb), & \ldots & (ff) \end{vmatrix} \begin{vmatrix} (bb), & \ldots & (bf), & (bn) \\ \cdot & \cdot & \cdot & \cdot \\ (fb), & \ldots & (ff), & (fn) \\ (nb), & \ldots & (nf), & (nn) \end{vmatrix}$$

$$- \begin{vmatrix} (ba), & (bb), & \ldots & (bf) \\ \cdot & \cdot & \cdot & \cdot \\ (fa), & (fb), & \ldots & (ff) \\ (na), & (nb), & \ldots & (nf) \end{vmatrix} \begin{vmatrix} (ab), & \ldots & (af), & (an) \\ (bb), & \ldots & (bf), & (bn) \\ \cdot & \cdot & \cdot & \cdot \\ (fb), & \ldots & (ff), & (fn) \end{vmatrix},$$

in which the product of two determinants of $\mu-1$ and $\mu+1$ rows respectively is equated to the difference of the products of pairs of determinants each of which contains μ rows.

The value of $(m-\mu)\epsilon_x^2$ may therefore be written

$$\frac{\begin{vmatrix} (bb), & \ldots & (bf), & (bn) \\ \cdot & \cdot & \cdot & \cdot \\ (fb), & \ldots & (ff), & (fn) \\ (nb), & \ldots & (nf), & (nn) \end{vmatrix}}{\begin{vmatrix} (aa), & (ab), & \ldots & (af) \\ (ba), & (bb), & \ldots & (bf) \\ \cdot & \cdot & \cdot & \cdot \\ (fa), & (fb), & \ldots & (ff) \end{vmatrix}} - \frac{\begin{vmatrix} (ab), & \ldots & (af), & (an) \\ (bb), & \ldots & (bf), & (bn) \\ \cdot & \cdot & \cdot & \cdot \\ (fb), & \ldots & (ff), & (fn) \end{vmatrix}^2}{\begin{vmatrix} (aa), & (ab), & \ldots & (af) \\ (ba), & (bb), & \ldots & (bf) \\ \cdot & \cdot & \cdot & \cdot \\ (fa), & (fb), & \ldots & (ff) \end{vmatrix}^2},$$

the second term of which is the value of x_0^2.

1880MNRAS..41...18G

The first term may be written

$$\frac{\begin{vmatrix} (an), & (ab), & \dots & (af) \\ (bn), & (bb), & \dots & (bf) \\ \cdot & \cdot & \cdot & \cdot \\ (fn), & (fb), & \dots & (ff) \end{vmatrix}}{\begin{vmatrix} (aa), & (ab), & \dots & (af) \\ (ba), & (bb), & \dots & (bf) \\ \cdot & \cdot & \cdot & \cdot \\ (fa), & (fb), & \dots & (ff) \end{vmatrix}} \cdot \frac{\begin{vmatrix} (nn), & (nb), & \dots & (nf) \\ (bn), & (bb), & \dots & (bf) \\ \cdot & \cdot & \cdot & \cdot \\ (fn), & (fb), & \dots & (ff) \end{vmatrix}}{\begin{vmatrix} (na), & (nb), & \dots & (nf) \\ (ba), & (bb), & \dots & (bf) \\ \cdot & \cdot & \cdot & \cdot \\ (fa), & (fb), & \dots & (ff) \end{vmatrix}},$$

and therefore is equal to $x_0\xi$, where ξ is the value of x given by the equations

$$\left.\begin{array}{l} (na)x+(nb)y \;\dots\; +(nf)t = (nn), \\ (ba)x+(bb)y \;\dots\; +(bf)t = (bn), \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ (fa)x+(fb)y \;\dots\; +(ff)t = (fn), \end{array}\right\}$$

§ 28. Thus we see that

$$\epsilon_x^2 = \frac{x_0(\xi - x_0)}{m-\mu},$$

$$\epsilon_y^2 = \frac{y_0(\eta - y_0)}{m-\mu},$$

$$\epsilon_z^2 = \text{\&c.},$$

where x_0, y_0, . . . are the most probable values of x, y, . . . , given by the normal equations, and ξ, η, ζ, . . . are the values of x, y, z, . . . given by the respective systems of equations,
for ξ

$$\left.\begin{array}{l} (na)x+(nb)y \;\dots\; +(nf)t = (nn), \\ (ba)x+(bb)y \;\dots\; +(bf)t = (bn), \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ (fa)x+(fb)y \;\dots\; +(ff)t = (fn), \end{array}\right\},$$

for η,

$$\left.\begin{array}{l} (aa)x+(ab)y \;\dots\; +(af)t = (an), \\ (na)x+(nb)y \;\dots\; +(nf)t = (nn), \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ (fa)x+(fb)y \;\dots\; +(ff)t = (fn), \end{array}\right\},$$

for ζ, &c.

Each of these systems of equations only differs from the system of normal equation by the substitution of

$$(na)x+(nb)y \;\dots\; +(nf)t = (nn), \quad \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

1880MNRAS..41...18G

for one of the equations, viz. in the ξ-system (1), replaces the first equation

$$(aa)x+(ab)y\ \ldots\ +(af)t=(an)\,;$$

in the η-system, it replaces the second equation

$$(ba)x+(bb)y\ \ldots\ +(bf)t=(bn),$$

and so on.

These forms of the values of $\epsilon_x{}^2$, $\epsilon_y{}^2$, &c., are, so far as I know, new.

§ 29. Applying the formulæ of the last section to the numerical example in § 20, viz.:—

$$\left.\begin{aligned} x-\ y+2z&=\ 3,\\ 3x+2y-5z&=\ 5,\\ 4x+\ y+4z&=21,\\ -\ x+3y+3z&=14,\end{aligned}\right\},$$

the normal equations are

$$\left.\begin{aligned} 27x+\ 6y\qquad\ &=\ 88,\\ 6x+15y+\ z&=\ 70,\\ y+54z&=107,\end{aligned}\right\}\ \ldots\ldots\ldots\ldots \text{(i)},$$

ξ is the value of x given by

$$\left.\begin{aligned} 88x+70y+107z&=671,\\ 6x+15y+\ z&=\ 70,\\ y+\ 54z&=107,\end{aligned}\right\}\ \ldots\ldots\ldots\ldots \text{(ii)},$$

η is the value of y given by

$$\left.\begin{aligned} 27x+\ 6y\qquad\ &=\ 88,\\ 88x+70y+107z&=671\\ y+\ 54z&=107,\end{aligned}\right\}\ \ldots\ldots\ldots\ldots \text{(iii)},$$

and ζ is the value of z given by

$$\left.\begin{aligned} 27x+\ 6y\qquad\ &=\ 88,\\ 6x+15y+\ z&=\ 70,\\ 88x+70y+107z&=671,\end{aligned}\right\}\ \ldots\ldots\ldots\ldots \text{(iv)},$$

The values of x_0, y_0, z_0 given by (i) are

$$x_0=\frac{49154}{19899},\quad y_0=\frac{2617}{737},\quad z_0=\frac{12707}{6633},$$

and solving (ii), (iii), (iv) we find

$$\xi=\frac{60742}{24577},\quad \eta=\frac{9297}{2617},\quad \zeta=\frac{24353}{12707},$$

1880MNRAS..41...18G

and therefore

$$\epsilon_x^2 = \frac{49154}{19899}\left(\frac{60742}{24577} - \frac{49154}{19899}\right),$$

$$\epsilon_y^2 = \frac{2617}{737}\left(\frac{9297}{2617} - \frac{2617}{737}\right),$$

$$\epsilon_z^2 = \frac{12707}{6633}\left(\frac{24353}{12707} - \frac{12707}{6633}\right),$$

whence

$$\epsilon_x^2 = \frac{1294400}{(19899)^2}, \quad \epsilon_y^2 = \frac{3200}{(737)^2}, \quad \epsilon_z^2 = \frac{65600}{(6633)^2},$$

which agree with the values given by the ordinary process; for

$$(vv) = \frac{1600}{19899},$$

and, if p_x, p_y, p_z denote the weights of x_o, y_o, z_o,

$$p_x = \frac{19899}{809}, \quad p_y = \frac{737}{54}, \quad p_z = \frac{6633}{122},$$

whence

$$\epsilon_x^2 = \frac{(vv)}{p_x} = \frac{1600}{19899} \times \frac{809}{19899} = \frac{1294400}{(19899)^2},$$

$$\epsilon_y^2 = \frac{(vv)}{p_y} = \frac{1600}{19899} \times \frac{54}{737} = \frac{3200}{(737)^2},$$

$$\epsilon_z^2 = \frac{(vv)}{p_z} = \frac{1600}{19899} \times \frac{123}{6633} = \frac{65600}{(6633)^2}.$$

§ 30. At first sight it would appear that $\epsilon_x = 0$, $\epsilon_y = 0$, &c., if $x_o = 0$, $y_o = 0$, &c., respectively: this is not, however, the case, for if, say, $x_o = 0$, ξ is infinite. This is evident, since x_o can only vanish if

$$\begin{vmatrix} (an), & (ab), & \ldots & (af) \\ (bn), & (bb), & \ldots & (bf) \\ \cdot & \cdot & \cdot & \cdot \\ (fn), & (fb) & \ldots & (ff) \end{vmatrix} = 0;$$

this determinant forms the denominator of ξ, and the numerator of ξ, by § 16, is expressible as a sum of squares, and cannot in general vanish.

When $x_o = 0$, the product $x_o \xi$ is to be replaced by its value

$$\begin{vmatrix} (nn), & (nb), & \ldots & (nf) \\ (bn), & (bb), & \ldots & (bf) \\ \cdot & \cdot & \cdot & \cdot \\ (fn), & (fb), & \ldots & (ff) \end{vmatrix} \div \begin{vmatrix} (aa), & (ab), & \ldots & (af) \\ (ba), & (bb), & \ldots & (bf) \\ \cdot & \cdot & \cdot & \cdot \\ (fa), & (fb), & \ldots & (ff) \end{vmatrix}$$

and if $y_o = 0$, &c., the products $y_o \eta$, &c., are to be similarly replaced.

1880MNRAS..41...18G

§ 31. If $\xi = x_0$, or $\eta = y_0$, &c., then in general ϵ_x, ϵ_y, &c., vanish. For since

$$(vv) = -(na)x_0 - (nb)y_0 - \ \ldots \ + (nn),$$

we may regard x_0 as the value of x given by the equations

$$\left.\begin{array}{l} (na)x + (nb)y \ \ldots \ + (nf)t = (nn) - (vv), \\ (ba)x + (bb)y \ \ldots \ + (bf)t = (bn), \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ (fa)x + (fb)y \ \ldots \ + (ff)t = (fn), \end{array}\right\},$$

viz.,

$$x_0 = \frac{1}{\nabla} \begin{vmatrix} (nn) - (vv), & (nb), & \ldots & (nf) \\ (bn), & (bb), & \ldots & (bf) \\ \cdot & \cdot & \cdot & \cdot \\ (fn), & (fb), & \ldots & (ff) \end{vmatrix},$$

and

$$\xi = \frac{1}{\nabla} \begin{vmatrix} (nn), & (nb), & \ldots & (nf) \\ (bn), & (bb), & \ldots & (bf) \\ \cdot & \cdot & \cdot & \cdot \\ (fn), & (fb), & \ldots & (ff) \end{vmatrix},$$

so that if $\xi = x_0$,

$$(vv) \begin{vmatrix} (bb), & (bc), & \ldots & (bf) \\ (cb), & (cc), & \ldots & (cf) \\ \cdot & \cdot & \cdot & \cdot \\ (fb), & (fc), & \ldots & (ff) \end{vmatrix} = 0,$$

the expression on the left-hand side of this equation, when multiplied by ∇, being in fact the numerator of ϵ_x^2 in § 14. Thus, if $\xi = x_0$, then, unless the second factor vanishes, $(vv) = 0$ and therefore $\epsilon_y = 0$, $\epsilon_z = 0$, &c., and $\eta = y_0$, $\zeta = z_0$, &c. If the second factor vanishes, since this determinant is a sum of squares, we must have

$$\left\| \begin{array}{ccccc} b_1, & b_2, & b_3 & \ldots & b_m \\ c_1, & c_2, & c_3, & \ldots & c_m \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f_1, & f_2, & f_3, & \ldots & f_m \end{array} \right\| = 0,$$

and these conditions imply that the equations of condition do not involve the unknowns $x, y \ldots t$ independently, *i.e.* the unknowns so enter into the m equations of condition that no set of μ equations, selected from them, affords determinate values of $x, y, \ldots t$.

1880MNRAS..41...18G

As mentioned at the end of § 24, if

$$\begin{vmatrix} (bb), & (bc), & \ldots & (bf) \\ (cb), & (cc), & \ldots & (cf) \\ . & . & . & . \\ (fb), & (fc), & \ldots & (ff) \end{vmatrix} = 0,$$

then $\nabla = 0$ and therefore also $(vv) = 0$, so that the values of x_o, y_o, &c., p_x, p_y, &c., ϵ_x, ϵ_y, &c., all assume the form $0 \div 0$.

Thus, if any one of the equations

$$\xi - x_0 = 0, \quad \eta - y_0 = 0, \text{ \&c.},$$

is satisfied, in general all the others are satisfied, and ϵ_x, ϵ_y, &c., all vanish. The case of exception is when the equations of condition are not of a form which suffices to determine all the unknowns, and then the whole process of solution becomes illusory.

Note on Hansen's Value of the Semi-diameter of the Moon.
By E. Neison, Esq.

In a Note in the *Monthly Notices* for November 1879 I have stated that, taking 57′ 0″ for the value adopted by Hansen for the constant of parallax, the mean semi-diameter of the Moon, employed by him in his *Tables de la Lune*, is 15′ 33″·47. This result follows from the data given by Hansen in the Introduction to his Tables; though it is not very easy from this source alone to determine the real value of the constants which he has adopted.

In the *Monthly Notices* for June 1880 (page 482), Prof. Adams points out that the mean value of the sine of the equatorial horizontal parallax employed by Hansen in his tables is really 57′ 2″·08. This it seems was pointed out in 1837 by Hansen himsef in the *Astronomische Nachrichten*, vol. xvii. page 299, in a paper which I had not seen until my attention was directed to it by Prof. Adams. The mean semi-diameter of the Moon employed by Hansen in his *Tables de la Lune*, therefore, is 15′ 34′ ·08, or is 0″·61 greater than stated in my previous Note. All the values given therefore in this Note must be increased by 0″·61, but this correction will not affect the accuracy of the conclusions drawn therein as to the apparent value of the lunar semi-diameter in different instruments. The empirical formula will be: Moon's semi-diameter=

$$15' \; 33'{\cdot}98 + 4''{\cdot}10 \div (1 + 0{\cdot}70 \times \text{aperture in inches}).$$